CHAPTER–10

VECTORS

POINTS TO REMEMBER

- A quantity that has magnitude as well as direction is called a vector. It is denoted by a directed line segment.

- Two or more vectors which are parallel to same line are called collinear vectors.

- Position vector of a point P(a, b, c) w.r.t. origin (0, 0, 0) is denoted by $\overline{OP}$ where $\overline{\mathrm{OP}} = \mathrm{a}\hat{\imath} + \mathrm{b}\hat{\jmath} + \mathrm{c}\hat{\mathrm{k}}$ and $|\overline{OP}| = \sqrt{\mathrm{a}^2 + \mathrm{b}^2 + \mathrm{c}^2}$.

- If $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be any two points in space, then

  $\overrightarrow{AB} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$and

  $|\overrightarrow{AB}| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z)^2}$

- Any vector $\vec{a}$ is called unit vector if $|\vec{a}| = 1$ It is denoted by $\hat{a}$

- If two vectors $\vec{a}$ and $\vec{b}$ are represented in magnitude and direction by the two sides of a triangle in order, then their sum $\vec{a} + \vec{b}$ is represented in magnitude and direction by third side of a triangle taken in opposite order. This is called triangle law of addition of vectors.

- If $\vec{a}$ is any vector and λ is a scalar, then $\lambda\,\vec{a}$ is vector collinear with $\vec{a}$ and $|\lambda\,\vec{a}| = |\lambda||\vec{a}|$.

- If $\vec{a}$ and $\vec{b}$ are two collinear vectors, then $\vec{a} = \lambda\,\vec{b}$ where λ is some scalar.

- Any vector $\vec{a}$ can be written as $\vec{a} = |\vec{a}|\hat{a}$ where $\hat{a}$ is a unit vector in the direction of $\vec{a}$.

- If $\vec{a}$ and $\vec{b}$ be the position vectors of points A and B, and C is any point which divides $\overrightarrow{AB}$ in ratio m:n internally then position vector $\vec{c}$ of point C is given as $\vec{c} = \frac{m\vec{b}+n\vec{a}}{m+n}$. If C divides $\overrightarrow{AB}$ in ratio m:n externally, then $\vec{c} = \frac{m\vec{b}-n\vec{a}}{m-n}$.

- The angles α, β and γ made by $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$ with positive direction of x, y and z-axis are called angles and cosines of these angles are called direction cosines of $\vec{r}$ usually denoted as $l = \cos\alpha$, $m = \cos\beta$, $n = \cos\gamma$

  Also $l = \frac{a}{|\vec{r}|}, m = \frac{b}{|\vec{r}|}, n = \frac{c}{|\vec{r}|}$ and $l^2 + m^2 + n^2 = 1$

- The numbers a, b, c proportional to l, m, n are called direction ratios.

- Scalar product or dot product of two vectors $\vec{a}$ and $\vec{b}$ is denoted as $\vec{a}.\vec{b}$ and is defined as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}| \cos\theta$, θ is the angle between $\vec{a}$ and $\vec{b}$.$(0 \le \theta \le \pi)$.

- Dot product of two vectors is commutative i.e. $\vec{a}.\vec{b} = \vec{b}.\vec{a}$

- $\vec{a}.\vec{b} = 0 \Leftrightarrow \vec{a} = \vec{o}, \vec{b} = \vec{o}$ or $\vec{a} \perp \vec{b}$.

- $\vec{a}.\vec{a} = |\vec{a}|^2$, so $\hat{i}\ \hat{i} = \hat{j}\ \hat{j} = \hat{k}\ k = 1$

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$, then

$$\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

- Projection of $\vec{a}$ on $\vec{b} = \left|\frac{\vec{a}.\vec{b}}{|\vec{b}|}\right|$ and

  Projection vector of $\vec{a}$ along $\vec{b} = \left(\frac{(\vec{a}.\vec{b})}{|\vec{b}|}\right)\hat{b}$.

- Cross product or vector product of two vectors $\vec{a}$ and $\vec{b}$ is denoted as $\vec{a} \times \vec{b}$ and is defined as $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin\theta\, \hat{n}$. were θ is the angle between $\vec{a}$ and $\vec{b}$.$(0 \le \theta \le \pi)$. And $\hat{n}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$ such that $\vec{a}\,.\,\vec{b}$ and $\hat{n}$ from a right handed system.

- Cross product of two vectors is not commutative i.e., $\vec{a} \times \vec{b} \neq \vec{b} \times \vec{a}$, but $\vec{a} \times \vec{b} = -(\vec{b} \times \vec{a})$.

- $\vec{a} \times \vec{b} = \vec{0} \Leftrightarrow \vec{a} = \vec{0}, \vec{b} = \vec{0}$ or $\vec{a} \| \vec{b}$.

- $\hat{\imath} \times \hat{\imath} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k} = \vec{0}$.

- $\hat{\imath} \times \hat{\jmath} = \hat{k}, \hat{\jmath} \times \hat{k} = \hat{\imath}, \hat{k} \times \hat{\imath} = \hat{\jmath}$ and $\hat{\jmath} \times \hat{\imath} = -\hat{k}, \hat{k} \times \hat{\jmath} = -\hat{\imath}, \hat{\imath} \times \hat{k} = -\hat{\jmath}$

- If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

- Unit vector perpendicular to both $\vec{a}$ and $\vec{b} = \pm\left(\frac{(\vec{a} \times \vec{b})}{|\vec{a} \times \vec{b}|}\right)$.

- $|\vec{a} \times \vec{b}|$ is the area of parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b}$

- $\frac{1}{2}|\vec{a} \times \vec{b}|$ is the area of parallelogram where diagonals are $\vec{a}$ and $\vec{b}$.

- If $\vec{a}, \vec{b}$ and $\vec{c}$ form a triangle, then area of the triangle

- $= \frac{1}{2}|\vec{a} \times \vec{b}| = \frac{1}{2}|\vec{b} \times \vec{c}| = \frac{1}{2}|\vec{c} \times \vec{a}|$.

- Scalar triple product of three vector $\vec{a}, \vec{b}$ and $\vec{c}$ is defined as $\vec{a}.(\vec{b} \times \vec{c})$ and is denotes as $[\vec{a}\vec{b}\,\vec{c}]$

- Geometrically, absolute value of scalar triple product $[\vec{a}\vec{b}\vec{c}]$ represents volume of a parallelopiped whose coterminous edges are $\vec{a}\,\vec{b}$ and $\vec{c}$.

- $\vec{a}\ \vec{b}\ \vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$

- $[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}]$

- If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}, \vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ and $\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then

$$[\vec{a}\vec{b}\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- Then scalar triple product of three vectors is zero if any two of them are same or collinear.

Very Short Answer Type Questions (1 MARK)

1. If $\overrightarrow{AB} = 3\hat{i} + 2\hat{j} - \hat{k}$ and the coordinate of A are (4,1,1), then find the coordinates of B.

2. Let $\vec{a} = -2\hat{i} + \hat{j}$, $\vec{b} = \hat{i} + 2\hat{j}$ and $\vec{c} = 4\hat{i} + 3\hat{j}$. Find the values of x and y such that $\vec{c} = x\vec{a} + y\vec{b}$.

3. Find a unit vector in the direction of the resultant of the vectors $\hat{i} - \hat{j} + 3\hat{k}$, $2\hat{i} + \hat{j} - 2\hat{k}$ and $\hat{i} + 2\hat{j} - 2\hat{k}$.

4. Find a vector of magnitude of 5 units parallel to the resultant of vector $\vec{a} = 2\hat{\imath} + 3\hat{\jmath} + \hat{k}$ and $\vec{b} = (\hat{\imath} - 2\hat{\jmath} - \hat{k})$

5. For what value of λ are the vector $\vec{a}\ and\ \vec{b}$ perpendicular to each other? Where $\vec{a} = \lambda\hat{\imath} + 2\hat{\jmath} + \hat{k}$ and $\vec{b} = 5\hat{\imath} - 9\hat{\jmath} + 2\hat{k}$

6. Write the value of p for which $\vec{a} = 3\hat{\imath} + 2\hat{\jmath} + 9\hat{k}$ and $\vec{b} = \hat{\imath} + p\hat{\jmath} + 3\hat{k}$ are parallel vectors.

7. For any two vectors $\vec{a}\ and\ \vec{b}$ write when $|\vec{a} + \vec{b}| = |\vec{a} - \vec{b}|$ holds.

8. Find the value of p if $(2\hat{\imath} + 6\hat{\jmath} + 27\hat{k}) \times (\hat{\imath} + 3\hat{\jmath} + p\hat{k}) = \vec{0}$

9. Evaluate: $\hat{\imath}.(\hat{\jmath} \times \hat{k}) + (\hat{\imath} \times \hat{k}).\hat{\jmath}$

10. If $\vec{a} = 2\hat{\imath} - 3\hat{\jmath},\ \vec{b} = \hat{\imath} + \hat{\jmath} - \hat{k}, \vec{c} = 3\hat{\imath} - \hat{k}$, find $[\vec{a}\vec{b}\vec{c}]$

11. If $\vec{a} = 5\hat{\imath} - 4\hat{\jmath} + \hat{k},\ \vec{b} = -4\hat{\imath} + 3\hat{\jmath} - 2\hat{k}$ and $\vec{c} = \hat{\imath} - 2\hat{\jmath} - 2\hat{k}$, then evaluate $\vec{c}.(\vec{a} \times \vec{b})$

12. Show that vector $\hat{\imath} + 3\hat{\jmath} + \hat{k},\ 2\hat{\imath} - \hat{\jmath} - \hat{k},\ 7\hat{\jmath} + 3\hat{k}$ are parallel to same plane.

13. Find a vector of magnitude 6 which is perpendicular to both the vectors $2\hat{\imath} - \hat{\jmath} + 2\hat{k}$ and $4\hat{\imath} - \hat{\jmath} + 3\hat{k}$.

14. If $\vec{a}\,.\vec{b} = 0$, then what can you say about $\vec{a}\ and\ \vec{b}$?

15. If $\vec{a}\ and\ \vec{b}$ are two vectors such that $|\vec{a} \times \vec{b}| = \vec{a}.\vec{b}$, then what is the angle between $\vec{a}\ and\ \vec{b}$?

16. Find the area of a parallelogram having diagonals $3\hat{\imath} + \hat{\jmath} - 2\hat{k}$ and $\hat{\imath} - 3\hat{\jmath} + 4\hat{k}$.

17. If $\hat{\imath}, \hat{\jmath}$ *and* $\hat{k}$ are three mutually perpendicular vectors, then find the value of $\hat{\jmath}.(\hat{k} \times \hat{\imath})$.

18. P and Q are two points with position vectors $3\vec{a} - 2\vec{b}$ and $\vec{a} + \vec{b}$ respectively. Write the position vector of a point R which divides the segment PQ in the ratio 2:1 externally.

19. Find λ when scalar projection of $\vec{a} = \lambda\hat{\imath} + \hat{\jmath} + 4\hat{k}$ on $\vec{b} = 2\hat{\imath} + 6\hat{\jmath} + 3\hat{k}$ is 4 units.

20. Find "a" so that the vectors $\vec{p} = 3\hat{\imath} - 2\hat{\jmath}$ and $\vec{q} = 2\hat{\imath} + a\hat{\jmath}$ be orthogonal.

21. If $\vec{a} = \hat{\imath} - \hat{\jmath} + \hat{k}$, $\vec{b} = 2\hat{\imath} + \hat{\jmath} - \hat{k}$ and $\vec{c} = \lambda\hat{\imath} - \hat{\jmath} + \lambda\hat{k}$ are coplanar, find the value of λ.

22. What is the point of trisection of PQ nearer to P if positions of P and Q are $3\hat{\imath} + 3\hat{\jmath} - 4\hat{k}$ and $9\hat{\imath} + 8\hat{\jmath} - 10\hat{k}$ respectively?

23. What is the angle between $\vec{a}$ and $\vec{b}$, if $\vec{a}.\vec{b} = 3$ and $|\vec{a} \times \vec{b}| = 3\sqrt{3}$.

SHORT ANSWER TYPE QUESTIONS (2 MARKS)

Q.1. A vector $\vec{r}$ is inclined to x – axis at 45° and y-axis at 60° if $|\vec{r}|$ = 8 units. find $\vec{r}$.

Q.2. if $|\vec{a}+\vec{b}|$ = 60, $|\vec{a}-\vec{b}|$ = 40 and $\vec{b}$ = 46 find $|\vec{a}|$

Q.3. Write the projection of $\vec{b}+\vec{c}$ on $\vec{a}$ where

$\vec{a} = 2\hat{i} - 2\hat{j} + \hat{k}, \vec{b} = \hat{i} + 2\hat{j} - 2\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$

Q.4. If the points (–1, –1, 2), (2, m, 5) and (3, 11, 6) are collinear, find the value of *m*.

Q.5. For any three vectors $\vec{a}, \vec{b}$ and $\vec{c}$ write value of the following.

$\vec{a} \times (\vec{b} + \vec{c}) + \vec{b} \times (\vec{c} + \vec{a}) + \vec{c} \times (\vec{a} + \vec{b})$

Q.6. If $(\vec{a}+\vec{b})^2 + (\vec{a}.\vec{b})^2$ = 144 and | $\vec{a}$ | = 4. Find the value of | $\vec{b}$ |.

Q.7. If for any two vectors $\vec{a}$ and $\vec{b}$,

$(\vec{a} + \vec{b})^2 + (\vec{a} - \vec{b})^2 = \lambda\left[(\vec{a})^2 + (\vec{b})^2\right]$ then write the value of λ .

Q.8. if $\vec{a}, \vec{b}$ are two vectors such that $|(\vec{a}+\vec{b})|$ = | $\vec{a}$ | then prove that 2 $\vec{a}$ + $\vec{b}$ is perpendicular to $\vec{b}$.

Q.9. Show that vectors $\vec{a} = 3\hat{i} - 2\hat{j} + \hat{k}$

$\vec{b} = \hat{i} - 3\hat{j} + 5\hat{k}, \vec{c} = 2\hat{i} + \hat{j} - 4\hat{k}$ form a right angle triangle.

Q.10. If $\vec{a}, \vec{b}, \vec{c}$ are three vectors such that $\vec{a}+\vec{b}+\vec{c}$ = 0 and | $\vec{a}$ | = 5, | $\vec{b}$ | = 12, | $\vec{c}$ | = 13, then find $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$

Q.11. The two vectors $\hat{i} + \hat{j}$ and $3\hat{i} - \hat{j} + 4\hat{k}$ represents the two sides AB and AC respectively of Δ ABC, find the length of median through A.

Short Answer Type Questions (4 Marks)

1. The points A,B and C with position vectors $3\hat{\imath} - y\hat{\jmath} + 2\hat{k}$, $5\hat{\imath} - \hat{\jmath} + \hat{k}$ and $3x\hat{\imath} + 3\hat{\jmath} - \hat{k}$ are collinear. Find the values of x and y and also the ratio in which the point B divides AC.

2. If sum of two unit vectors is a unit vector, prove that the magnitude of their difference is $\sqrt{3}$.

3. Let $\vec{a} = 4\hat{\imath} + 5\hat{\jmath} - \hat{k}$, $\vec{b} = \hat{\imath} - 4\hat{\jmath} + 5\hat{k}$ and $\vec{c} = 3\hat{\imath} + \hat{\jmath} - \hat{k}$. Find a vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and satisfying $\vec{d}.\vec{c} = 21$

4. If $\hat{a}$ and $\hat{b}$ are unit vectors inclined at an angle θ then proved that

   (i) $\cos\frac{\theta}{2} = \frac{1}{2}|\hat{a} + \hat{b}|$

   (ii) $\tan\frac{\theta}{2} = \left|\frac{\hat{a} - \hat{b}}{\hat{a} - \hat{b}}\right|$

5. If $\vec{a},\vec{b},\vec{c}$ are three mutually perpendicular vectors of equal magnitude. Prove that $\vec{a}+\vec{b}+\vec{c}$ is equally in clined with vectors $\vec{a},\vec{b}$ and $\vec{c}$. Also find angles.

6. For any vector $\vec{a}$ prove that $|\vec{a} \times \hat{\imath}|^2 + |\vec{a} \times \hat{\jmath}|^2 + |\vec{a} \times \hat{k}|^2 = 2|\vec{a}|^2$

7. Show that $(\vec{a} \times \vec{b})^2 = |\vec{a}|^2|\vec{b}|^2 - (\vec{a}.\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{a}.\vec{b} & \vec{b}.\vec{b} \end{vmatrix}$

8. If $\vec{a}, \vec{b}$ and $\vec{c}$ are the position vectors of vertices A,B,C of a Δ ABC, show that the area of triangle ABC is $\frac{1}{2}|\vec{a} \times \vec{b} + \vec{b} \times \vec{c} + \vec{c} \times \vec{a}|$. Deduce the condition for points $\vec{a}, \vec{b}$ and $\vec{c}$ to be collinear.

9. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be unit vectors such that $\vec{a}.\vec{b} = \vec{a}.\vec{c} = 0$ and the angle between b and c is $\pi/6$, prove that $\vec{a} = \pm 2(\vec{b} \times \vec{c})$.

10. If $\vec{a}, \vec{b}$ and $\vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$, then prove that $\vec{a} \times \vec{b} = \vec{b} \times \vec{c} = \vec{c} \times \vec{a}$.

11. If $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{c} = \hat{\jmath} - \hat{k}$ are given vectors, then find a vector $\vec{b}$ satisfying the equations $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{a}.\vec{b} = 3$.

12. Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three non zero vectors such that $\vec{c}$ is a unit vector perpendicular to both $\vec{a}$ and $\vec{b}$. if the angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$, prove that $[\vec{a}\vec{b}\vec{c}]^2 = \frac{1}{4}|\vec{a}|^2|\vec{b}|^2$

13. If the vectors $\vec{\alpha} = a\hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{\beta} = \hat{\imath} + b\hat{\jmath} + \hat{k}$ and $\vec{\gamma} = \hat{\imath} + \hat{\jmath} + c\hat{k}$ are coplanar, then prove that $\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c} = 1$ where $a \neq 1, b \neq 1$ and $c \neq 1$

14. Find the altitude of a parallelepiped determined by the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ if the base is taken as parallelogram determined by $\vec{a}$ and $\vec{b}$ and if $\vec{a} = \hat{\imath} + \hat{\jmath} + \hat{k}$, $\vec{b} = 2\hat{\imath} + 4\hat{\jmath} - \hat{k}$ and $\vec{c} = \hat{\imath} + \hat{\jmath} + 3\hat{k}$.

15. Prove that the four points $(4\hat{i} + 5\hat{j} + \hat{k}), -(\hat{j} + \hat{k}), (3\hat{i} + 9\hat{j} + 4\hat{k})$ and $4(-\hat{i} + \hat{j} + \hat{k})$ are coplanar.

16. If $|\vec{a}| = 3$, $|\vec{b}| = 4$ and $|\vec{c}| = 5$ such that each is perpendicular to sum of the other two, find $|\vec{a} + \vec{b} + \vec{c}|$

17. Decompose the vector $6\hat{\imath} - 3\hat{\jmath} - 6\hat{k}$ into vectors which are parallel and perpendicular to the vector $\hat{\imath} + \hat{\jmath} + \hat{k}$.

18. If $\vec{a}, \vec{b}$ *and* $\vec{c}$ are vectors such that $\vec{a}.\vec{b} = \vec{a}.\vec{c}$, $\vec{a} \times \vec{b} = \vec{a} \times \vec{c}$, $a \neq 0$, then show that $\vec{b} = \vec{c}$.

19. If $\vec{a}, \vec{b}$ *and* $\vec{c}$ are three non zero vectors such that $\vec{a} \times \vec{b} = \vec{c}$ and $\vec{b} \times \vec{c} = \vec{a}$. Prove that $\vec{a}, \vec{b}$ *and* $\vec{c}$ are mutually at right angles and $|\vec{b}| = 1$ and $|\vec{c}| = |\vec{a}|$

20. Simplify $[\vec{a} - \vec{b},\ \vec{b} - \vec{c},\ \vec{c} - \vec{a}]$

21. If $[\vec{a}\vec{b}\vec{c}] = 2$, find the volume of the parallelepiped whose co-terminus edges are $2\vec{a} + \vec{b},\ 2\vec{b} + \vec{c},\ 2\vec{c} + \vec{a}$.

22. If $\vec{a}, \vec{b}$ *and* $\vec{c}$ are three vectors such that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 3$, $|\vec{b}| = 5,\ |\vec{c}| = 7,$ find the angle between $\vec{a}$ *and* $\vec{b}$.

23. The magnitude of the vector product of the vector $\hat{\imath} + \hat{\jmath} + \hat{k}$ with a unit vector along the sum of the vector $2\hat{i} + 4\hat{j} - 5\hat{k}$ and $\lambda\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$ is equal to $\sqrt{2}$. Find the value of λ.

24. If $\vec{a} \times \vec{b} = \vec{c} \times \vec{d}$ and $\vec{a} \times \vec{c} = \vec{b} \times \vec{d}$, prove that $(\vec{a} - \vec{d})$ is parallel to $(\vec{b} - \vec{c})$, where $\vec{a} \neq \vec{d}$ and $\vec{b} \neq \vec{c}$.

25. Find a vector of magnitude $\sqrt{171}$ which is perpendicular to both of the vectors $\vec{a} = \hat{i} + 2\hat{j} - 3\hat{k}$ and $\vec{b} = 3\hat{i} - \hat{j} + 2\hat{k}$.

26. If a is a nonzero real number, prove that the vectors
$\vec{\alpha} = a\hat{i} + 2a\hat{j} - 3a\hat{k}$, $\vec{\beta} = (2a+1)\hat{i} + (2a+3)\hat{j} + (a+1)\hat{k}$
and $\vec{\gamma} = (3a+5)\hat{i} + (a+5)\hat{j} + (a+2)\hat{k}$ are never coplanar.

27. If $\vec{\alpha} = 3\hat{i} - \hat{j}$ and $\vec{\beta} = 2\hat{i} + \hat{j} + 3\hat{k}$ then express $\vec{\beta}$ in the form of $\vec{\beta} = \vec{\beta}_1 + \vec{\beta}_2$, where $\vec{\beta}_1$ is parallel to $\vec{\alpha}$ and $\vec{\beta}_2$ is perpendicular to $\vec{\alpha}$.

28. Find a unit vector perpendicular to plane ABC, when position vectors of A,B,C are $\widehat{3\imath} - \hat{\jmath} + 2\hat{k}$, $\hat{\imath} - \hat{\jmath} - 3\hat{k}$ and $4\hat{\imath} - \widehat{3\jmath} + \hat{k}$ respectively.

29. Find a unit vector in XY plane which makes an angle 45° with the vector $\hat{\imath} + \hat{\jmath}$ at angle of 60° with the vector $3\hat{\imath} - 4\hat{\jmath}$.

30. Suppose $\vec{a} = \lambda\hat{\imath} - 7\hat{\jmath} + 3\hat{k}$, $\vec{b} = \lambda\hat{\imath} + \hat{\jmath} + 2\lambda\hat{k}$. If the angle between $\vec{a}$ and $\vec{b}$ is greater than 90°, then prove that λ satisfies the inequality$-7 < \lambda < 1$.

31. Let $\vec{v} = \widehat{2\imath} + \hat{\jmath} - \hat{k}$ and $\vec{w} = \hat{\imath} + 3\hat{k}$. If u is a unit vector, then find the maximum value of he scalar triple products $\vec{u}, \vec{v}, \vec{w}$.

32. If $\vec{a} = \hat{\imath} - \hat{k}, \vec{b} = x\hat{\imath} + \hat{\jmath} + (1-x)\hat{k}$ and $\vec{c} = y\hat{\imath} + x\hat{\jmath} + (1+x-y)\hat{k}$ then prove that $[\vec{a}\vec{b}\vec{c}]$ depends upon neither x nor y.

33. A, b and c are distinct non negative numbers, if the vectors $a\hat{\imath} + a\hat{\jmath} + c\hat{k}$, $\hat{\imath} + \hat{k}$ and $c\hat{\imath} + c\hat{\jmath} + b\hat{k}$ lie in a plane, then prove that c is the geometric mean of a and b.

34. If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and vectors $(1, a, a^2), (1, b, b^2)$ *and* $(1, c, c^2)$ are non-coplanar, then find the value of abc. (Ans. = –1)

Answers

Very Short Answer

1. (7, 3, 0)

2. $x = -1,\ y = 2$

3. $\frac{1}{\sqrt{21}}\left(4\hat{i} + 2\hat{j} - \hat{k}\right)$

4. $\sqrt{\frac{5}{2}}(3\hat{i} + \hat{j})$

5. $\lambda = \frac{16}{5}$

6. $\frac{2}{3}$

7. $\vec{a}$ *and* $\vec{b}$ *are per perdicular*

8. $\frac{27}{2}$

9. 0

10. 4

11. –5

12. ....

13. $-2\hat{i} + 4\hat{j} + 4\hat{k}$

14. Either $\vec{a} = 0$ or $\vec{b} = 0$ or $\vec{a} \perp \vec{b}$

15. 45°

16. $5\sqrt{3}$ sq. Units

17. 1

18. $-\vec{a} + 4\vec{b}$

19. $\lambda = 5$

20. a = 3

21. $\lambda = 1$

22. $\left(5, \frac{14}{3}, -6\right)$

23. $\pi/3$

SHORT ANSWER TYPE [2 MARKS]

1. $4(\sqrt{2}\hat{i} + \hat{j} + \hat{k})$
2. 22
3. 2
4. $m = 8$
5. 0
6. 3
7. $\lambda = 2$
8. —
9. —

10. –169

11. $2\sqrt{2}$

Short Answer Type Answer (4 marks)

1. $x = 3,\ y = 3,\ 1:2$

3. $\vec{d} = 7\hat{\imath} - 7\hat{\jmath} - 7\hat{k}$

5. $\cos^{-1}\frac{1}{\sqrt{3}}$

11. $\vec{b} = \frac{5}{3}\hat{\imath} + \frac{2}{3}\hat{\jmath} + \frac{2}{3}\hat{k}$

14. $\frac{4}{\sqrt{38}}$ units

16. $5\sqrt{2}$

17. $(-\hat{\imath} - \hat{\jmath} - \hat{k}) + (7\hat{\imath} - 2\hat{\jmath} - 5\hat{k})$

20. 0

21. 18 cu. Units

22. 60°

23. $\lambda = 1$

25. $\hat{i} - 11\hat{j} - 7\hat{k}$

27. $\vec{\beta} = \left(\frac{3}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath}\right) + \left(\frac{1}{2}\hat{\imath} + \frac{3}{2}\hat{\jmath} + 3\hat{k}\right)$

28. $\frac{-1}{\sqrt{165}}(10\hat{\imath} + 7\hat{\jmath} - 4\hat{k})$

29. $\frac{13}{\sqrt{170}}\hat{i}+\frac{1}{\sqrt{170}}\hat{j}$

31. $\sqrt{59}$

34. -1

CHAPTER–11

THREE-DIMENSIONAL GEOMETRY

POINTS TO REMEMBER

- **Distance Formula:** Distance (d) between two points(x_1 , y_1 , z_1)and(x_2, y_2, z_2)

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

- **Section Formula:** line segment AB is divided by P (x, y, z) in ratio m:n

(a) Internally	(b) Externally
$\left(\frac{m\,x_2 + n\,x_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n}\right)$	$\left(\frac{mx_2 - nx_1}{m-n}, \frac{my_2 - ny_1}{m-n}, \frac{mz_2 - nz_1}{m-n}\right)$

- **Direction ratio** of a line through $(x_1, y_1, z_1)$and (x_2, y_2, z_2) are$x_2 - x_1, y_2 - y_1, z_2 - z_1$

- **Direction cosines** of a line having direction ratios as a, b, c are:

$$l = \pm \frac{a}{\sqrt{a^2+b^2+c^2}}, \quad m = \pm \frac{b}{\sqrt{a^2+b^2+c^2}}, \quad n = \pm \frac{c}{\sqrt{a^2+b^2+c^2}}$$

- **Equation of line in space:**

Vector form	Cartesian form
(i) Passing through point $\vec{a}$ and parallel to vector $\vec{b}$; $\vec{r} = \vec{a} + \lambda\vec{b}$	(i) Passing through point (x_1, y_1, z_1) and having direction ratios a, b, c;

	$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$
(ii)Passing through two points $\vec{a}$and$\vec{b}$; $\vec{r} = \vec{a} + \lambda\,(\vec{b} - \vec{a})$	(ii) Passing through two points $(x_1, y_1 z_1)$ and $(x_2, y_2 z_2)$; $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$

- **Angle between two lines:**

Vector form	Cartesian form
(i) For lines $\vec{r} = \overrightarrow{a_1} + \lambda\,\overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \mu\,\overrightarrow{b_2}$ $\cos\theta = \frac{\lvert\vec{b}_1 . \vec{b}_2\rvert}{\lvert\vec{b}_1\rvert\lvert\vec{b}_2\rvert}$	(ii) For lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ $\cos\theta$ $= \frac{\lvert a_1a_2 + b_1b_2 + c_1c_2\rvert}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$
(iii) Lines are perpendicular if $\vec{b}_1 . \vec{b}_2$ 0	(ii) Lines are perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$
(iv) Lines are parallel if $\vec{b}_1 = k\vec{b}_2$; $k \neq 0$	(i) Lines are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

- **Equation of plane:**

If p is length of perpendicular from origin to plane and $\hat{n}$ is unit vector normal to plane $\vec{r} . \hat{n} = p$	If p is length of perpendicular from origin to plane and l, m, n ared.c.s of normal to plane $lx + my + nz = p$

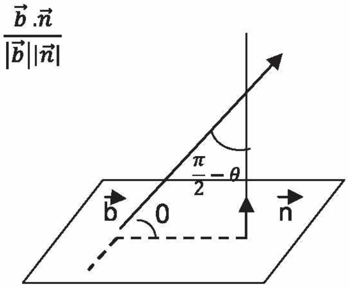
Passing through $\vec{a}$ and $\vec{n}$ is normal to plane :$(\vec{r} - \vec{a}).\vec{n} = 0$	Passing through (x_1, y_1, z_1) and a, b, c are d.r.s of normal to plane: $$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$
Passing through three non collinear points $\vec{a}, \vec{b}, \vec{c}$: $$(\vec{r} - \vec{a}).[(\vec{b} - \vec{a}) \times (\vec{c} - \vec{a})] = 0$$	Passing through three non collinear points$(x_1, y_1, z_1)(x_2, y_2, z_2)(x_3, y_3, z_3)$: $$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$
If a, b, c are intercepts on co-ordinate axes $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$	If x_1, y_1, z_1 are intercepts on coordinate axes $\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$
Plane passing through line of intersection of planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is $\vec{r}.(\vec{n}_1 + \lambda\vec{n}_2) = d_1 + \lambda d_2$ (λ=real no.)	Plane passing through the line of intersection of planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ is $$(a_1x + b_1y + c_1z + d_1) + \lambda(a_2x + b_2y + c_2z + d_2) = 0$$

- **Angle between planes:**

Angle θ between planes $\vec{r}.\vec{n}_1 = d_1$ and $\vec{r}.\vec{n}_2 = d_2$ is $$\cos\theta = \frac{\lvert\vec{n}_1 . \vec{n}_2\rvert}{\lvert\vec{n}_1\rvert\lvert\vec{n}_2\rvert}$$	Angle θ between planes $a_1x + b_1y + c_1z = d_1$ and $a_2x + b_2y + c_2z = d_2$ is $$\cos\theta = \frac{\lvert a_1a_2 + b_1b_2 + c_1c_2\rvert}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Planes are perpendicular iff $\vec{n}_1 \cdot \vec{n}_2 = 0$	Planes are perpendicular iff $a_1a_2 + b_1b_2 + c_1c_2 = 0$
Planes are parallel iff $\vec{n}_1 = \lambda \vec{n}_2$; $\lambda \neq 0$	Planes are parallel if $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

- **Angle between line and plane:**

Angle θ between line $\vec{r} = \vec{a} + \lambda\vec{b}$ and plane $\vec{r} \cdot \vec{n} = d$ is $\sin\theta = \cos(90^\circ - \theta)$ $= \frac{\vec{b} \cdot \vec{n}}{\lvert\vec{b}\rvert\lvert\vec{n}\rvert}$ 	Angle θ between line $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and plane $a_2x + b_2y + c_2z = d$ is $\sin\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$

- **Distance of a point from a plane**

The perpendicular distance p from the point P with position vector $\vec{a}$ to the plane $\vec{r} \cdot \vec{n} = d$ is given by $p = \frac{\lvert\vec{a} \cdot \vec{n} - d\rvert}{\lvert\vec{n}\rvert}$	The perpendicular distance p from the point P (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is given by $p = \frac{\lvert Ax_1 + By_1 + Cz_1 + D\rvert}{\sqrt{A^2 + B^2 + C^2}}$

- **Coplanarity**

Two lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ are coplanar iff $(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2) = 0$	Two lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ are coplanar iff $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$

- **Shortest distance between two skew lines**

The shortest distance between lines $\vec{r} = \vec{a}_1 + \lambda\vec{b}_1$ and $\vec{r} = \vec{a}_2 + \mu\vec{b}_2$ is $d = \left\lvert \frac{(\vec{a}_2 - \vec{a}_1).(\vec{b}_1 \times \vec{b}_2)}{\lvert \vec{b}_1 \times \vec{b}_2 \rvert} \right\rvert$	The shortest distance between $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$is $d = \frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{D}}$ Where $D = \{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2\}$

Very Short Answer Type Questions (1 Mark)

1. What is the distance of point (a, b, c) from x-axis?

2. What is the angle between the lines $2x = 3y = -z$ and $6x = -y = -4z$?

3. Write the equation of a line passing through (2, -3, 5) and parallel to line $\frac{x-1}{3} = \frac{y-2}{4} = \frac{z+1}{-1}$.

4. Write the equation of a line through (1, 2, 3) and parallel to $\bar{r}.(\hat{i} - \hat{j} + 3\hat{k}) = 5$.

5. What is the value of λ for which the lines $\frac{x-1}{2} = \frac{y-3}{5} = \frac{z-1}{\lambda}$ and $\frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{2}$ are perpendicular to each other?

6. Write line $\bar{r} = (\hat{i} - \hat{j}) + \lambda\,(2\hat{j} - \hat{k})$ into Cartesian form.

7. If the direction ratios of a line are 1, -2, 2 then what are the direction cosines of the line?

8. Find the angle between the planes $2x - 3y + 6z = 9$ and xy – plane.

9. Write equation of a line passing through (0, 1, 2) and equally inclined to co-ordinate axes.

10. What is the perpendicular distance of plane $2x - y + 3z = 10$ from origin?

11. What is the y-intercept of the plane $x - 5y + 7z = 10$?

12. What is the distance between the planes $2x + 2y - z + 2 = 0$ and $4x + 4y - 2z + 5 = 0$.

13. What is the equation of the plane which cuts off equal intercepts of unit length on the coordinate axes?

14. Are the planes $x + y - 2z + 4 = 0$ and $3x + 3y - 6z + 5 = 0$ intersecting?

15. What is the equation of the plane through the point (1, 4, -2) and parallel to the plane $-2x + y - 3z = 7$?

16. Write the vector equation of the plane which is at a distance of 8 units from the origin and is normal to the vector $(2\hat{\imath} + \hat{\jmath} + 2\hat{k})$.

17. What is equation of the plane if the foot of perpendicular from origin to this plane is (2, 3, 4)?

18. Find the angles between the planes $\vec{r}.(\hat{\imath} - 2\hat{\jmath} - 2\hat{k}) = 1$ and $\vec{r}.(3\hat{\imath} - 6\hat{\jmath} + 2\hat{k}) = 0$.

19. If O is origin OP = 3 with direction ratios proportional to -1, 2, -2 then what are the coordinates of P?

20. What is the distance between the line $\bar{r} = 2\hat{\imath} - 2\hat{\jmath} + 3\hat{k} + \lambda\,(\hat{\imath} + \hat{\jmath} + 4\hat{k})$ from the plane $\bar{r}\,.(-\hat{\imath} + 5\hat{\jmath} - \hat{k}) + 5 = 0$.

21. Write the line $2x = 3y = 4z$ in vector form.

22. The line $\frac{x-4}{1} = \frac{2y-4}{2} = \frac{k-z}{-2}$ lies exactly in the plane $2x - 4y + z = 7$. Find the value of k.

SHORT ANSWER TYPE QUESTIONS (2 MAKS EACH)

Q.23. What is the angle between the line $\frac{x+1}{3} = \frac{2y-1}{4} = \frac{2-z}{-4}$ and the plane 2x + y – 2z + 4 = 0

Q.24. Find the equation of a line passing though (2, 0, 5) and which is parallel to line 6x – 2 = 3y + 1 = 2z –2

Q.25. Find the equation of the plane passing through the points (2, 3, –4) and (1, –1, 3) and parallel to the x – axis.

Q.26. Find the distance between the planes 2x + 3y –4z + 5 = 0 and $\hat{r}.(4\hat{i} + 6\hat{j} - 8\hat{k}) = 11$

Q.27. The equation of a line are 5x – 3 = 15y + 7 = 3 – 10 z. Write the direction cosines of the line

Q.28. If a line makes angle α, β, γ with Co-ordinate axis then what is the value of $\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma$

Q.29. Find the equation of a line passing through the point (2, 0, 1) and parallel to the line whose equation is $\vec{r} = (2\lambda + 3)\hat{i} + (7\lambda - 1)\hat{j} + (-3\lambda + 2)\hat{k}$

Q.30. The plane 2x – 3y + 6 z – 11 = 0 makes an angle $\sin^{-1} \alpha$ with x – axis. Find the value of α.

Q.31. If 4x + 4y – cz = 0 is the equation of the plane passing through the origin that contains the line $\frac{x+5}{2} = \frac{y}{3} = \frac{z-7}{4}$, then find the value of c.

Q.32. Find the equation of the plane passing through the point (–2, 1, –3) and making equal intercept on the coordinate axes.

Q.33. Write the sum of intercepts cut off by the plane $\hat{r}.(2\hat{i}+\hat{j}-\hat{k})-5=0$ on the three axis.

Short Answer Type Questions (4 Marks)

34. Find the equation of a plane containing the points (0, –1, –1), (–4, 4, 4) and (4, 5, 1). Also show that (3, 9, 4) lies on that plane.

35. Find the equation of the plane which is perpendicular to the plane $\bar{r}.(5\hat{\imath}+3\hat{\jmath}+6\hat{k})+8=0$ and which is containing the line of intersection of the planes $\bar{r}.(\hat{\imath}+2\hat{\jmath}+3\hat{k})=4$ and $\bar{r}.(2\hat{\imath}+\hat{\jmath}-\hat{k})+5=0$.

36. Find the distance of the point (3, 4, 5) from the plane $x+y+z=2$ measured parallel to the line $2x=y=z$.

37. Find the equation of the plane passing through the intersection of two planes x + 2y + 3z – 5 = 0 and 3x – 2y – Z + 1 = 0 and cutting equal intercepts on x-axis and z-axis.

38. Find vector and Cartesian equation of a line passing through a point with position vector $2\hat{\imath}-\hat{\jmath}+\hat{k}$ and which is parallel to the line joining the points with position vectors$-\hat{\imath}+4\hat{\jmath}+\hat{k}$ and $\hat{\imath}+2\hat{\jmath}+2\hat{k}$.

39. Find the equation of the plane passing through the point (3, 4, 2) and (7, 0, 6) and is perpendicular to the plane $2x-5y=15$.

40. Find equation of plane through line of intersection of planes $\overline{r}\,.\,(2\hat{\imath} + 6\hat{\jmath}) + 12 = 0$ and $\overline{r}\,.\,(3\hat{\imath} - \hat{\jmath} + 4\hat{k}) = 0$ which is at a unit distance from origin.

41. Find the image of point (3, –2, 1) in the plane $3x - y + 4z = 2$.

42. Find image (reflection) of the point (7, 4, –3) in the line $\frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$.

43. Find equation of a plane passing through the points (2, –1, 0) and (3,–4, 5) and parallel to the line $2x = 3y = 4z$.

44. Find the distance of the point (–1, –5, –10) from the point of intersection of line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{2}$ and the plane $x - y + z = 5$.

45. Find the distance of the point (1, -2, 3) from the plane $x - y + z = 5$, measured parallel to the line $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$.

46. Find the equation of the plane passing through the intersection of two plane $3x - 4y + 5z = 10, 2x + 2y - 3z = 4$ and parallel to the line $x = 2y = 3z$.

47. Find the equation of the planes parallel to the plane $x - 2y + 2z - 3 = 0$ whose perpendicular distance from the point (1, 2, 3) is 1 unit.

48. Show that the lines line $\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}$ and line $\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$ intersect each other. Find the point of intersection.

49. Find the shortest distance between the lines:

$\bar{r} = \hat{\imath} + 2\hat{\jmath} + 3\hat{k} + \mu(2\hat{\imath} + 3\hat{\jmath} + 4\hat{k})$ and

$\bar{r} = (2\hat{\imath} + 4\hat{\jmath} + 5\hat{k}) + \lambda(3\hat{\imath} + 4\hat{\jmath} + 5\hat{k})$.

50. Find the distance of the point (-2, 3, -4) from the line $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$ measured parallel to the plane $4x + 12y - 3z + 1 = 0$.

51. Find the equation of plane passing through the point (-1, -1, 2) and perpendicular to each of the plane

$\bar{r}.(2\hat{\imath} + 3\hat{\jmath} - 3\hat{k}) = 2$ and $\bar{r}.(5\hat{\imath} - 4\hat{\jmath} + \hat{k}) = 6$

52. Find the equation of a plane passing through (-1, 3, 2) and parallel to each of the line $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ and $\frac{x+2}{-3} = \frac{y-1}{2} = \frac{z+1}{5}$

53. Show that the plane $\bar{r}.(\hat{\imath} - 3\hat{\jmath} + 5\hat{k}) = 7$ contains the line

$\bar{r} = (\hat{\imath} + 3\hat{\jmath} + 3\hat{k}) + \lambda(3\hat{\imath} + \hat{\jmath})$.

Long Answer Type Questions (6 Marks)

54. Check the co planarity of lines

$\bar{r} = (-3\hat{\imath} + \hat{\jmath} + 5\hat{k}) + \lambda(-3\hat{\imath} + \hat{\jmath} + 5\hat{k})$.

$$\bar{r} = (-\hat{\imath} + 2\hat{\jmath} + 5\hat{k}) + \mu(-\hat{\imath} + 2\hat{\jmath} + 5\hat{k})$$

If they are coplanar, find equation of the plane containing the lines.

55. Find shortest distance between the lines:

$\frac{x+1}{7} = \frac{y+1}{-6} = \frac{z+1}{1}$ and $\frac{x-3}{1} = \frac{5-y}{2} = \frac{z-7}{1}$

56. Find the shortest distance between the lines:

$\bar{r} = (1-\lambda)\hat{\imath} + (\lambda - 2)\hat{\jmath} + (3 - 2\lambda)\hat{k}$

$$\bar{r} = (\mu + 1)\hat{\imath} + (2\mu - 1)\hat{\jmath} - (2\mu + 1)\hat{k}$$

57. A variable plane is at a constant distance 3 p from the origin and meets the coordinates axes in A, B and C. If the centroid of $\Delta\, ABC$ is (α, β, γ), then show that $\alpha^{-2} + \beta^{-2} + \gamma^{-2} = p^{-2}$

58. A vector $\vec{n}$ of magnitude 8 units is inclined to x-axis at 45°, y axis at 60° and an acute angle with z-axis. If a plane passes through a point $(\sqrt{2}, -1, 1)$and is normal to $\vec{n}$, find its equation in vector form.

59. Find the foot of perpendicular from the point $2\hat{\imath} - \hat{\jmath} + 5\hat{k}$ on the line $\bar{r} = (11\hat{\imath} - 2\hat{\jmath} - 8\hat{k}) + \lambda(10\hat{\imath} - 4\hat{\jmath} - 11\hat{k})$. Also find the length of the perpendicular.

60. A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonal of a cube. Prove that $cos^2\alpha + cos^2\beta + cos^2\gamma + cos^2\delta = \frac{4}{3}$

61. Find the equation of the plane passing through the intersection of planes $2x + 3y - z = -1$ and $x + y - 2z + 3 = 0$ and perpendicular to the plane $3x - y - 2z = 4$. Also find the inclination of this plane with xy-plane.

62. Find the length and the equations of the line of shortest distance between the lines $\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7}$ and $\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5}$.

63. Show that $\frac{x-1}{2} = \frac{y+1}{3} = z$ and $\frac{x+1}{5} = \frac{y-2}{2}, z = 2$. do not intersect each other.

ANSWERS

3 DIMENSIONAL GEOMETRY 2 MARKS EACH

Q.23. 0° (line is parallel to plane)

Q.24. $\frac{x-2}{1} = \frac{y}{2} = \frac{z-5}{3}$

Q.25. 7y + 4z = 5

Q.26. $\frac{21}{2\sqrt{29}}$ units

Q.27. $\frac{6}{7}, \frac{2}{7}, \frac{-3}{7}$

Q.28. 2

Q.29. $\vec{r} = (2\hat{i} + \hat{k}) + \lambda(2\hat{i} + 7\hat{j} - \hat{k})$

Q.30. $\alpha = \frac{2}{7}$

Q.31. C = 5

Q.32. x + y + z = –4

Q.33. $\frac{5}{2}$

Answers

1. $\sqrt{b^2 + c^2}$
2. 90°
3. $\frac{x-2}{3} = \frac{y+3}{4} = \frac{z-5}{-1}$
4. $\bar{r} = (\hat{\imath} + 2\hat{\jmath} + 3\hat{k}) + \lambda(\hat{\imath} - \hat{\jmath} + 3\hat{k})$
5. $\lambda = 2$
6. $\frac{x-1}{0} = \frac{y+1}{2} = \frac{z}{-1}$
7. $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{2}{3}$
8. $\cos^{-1}(6/7)$
9. $\frac{x}{a} = \frac{y-1}{a} = \frac{z-2}{a}$, $a \in R - \{0\}$
10. $\frac{10}{\sqrt{14}}$
11. -2
12. $\frac{1}{6}$
13. $x + y + z = 1$
14. No
15. $-2x + y - 3z = 8$
16. $\bar{r} . (2\hat{\imath} + \hat{\jmath} + 2\hat{k}) = 24$
17. $2x + 3y + 4z = 29$
18. $\cos^{-1}\left(\frac{11}{21}\right)$
19. (–1, 2,–2)
20. $\frac{10}{3\sqrt{3}}$
21. $\vec{r} = \vec{0} + \lambda(6\hat{\imath} + 4\hat{\jmath} + 3\hat{k})$.

22. k = 7

34. $5x - 7y + 11z + 4 = 0$

35. $\bar{r}.(-51\hat{\imath} - 15\hat{\jmath} + 50\hat{k}) = 173$

36. 6 units

37. 5x + 2y + 5z – 9 = 0

38. $\bar{r} = (2\hat{\imath} - \hat{\jmath} + \hat{k}) + \lambda(2\hat{\imath} - 2\hat{\jmath} + \hat{k})$

and $\frac{x-2}{2} = \frac{y+1}{-2} = \frac{z-1}{1}$

39. $5x + 2y - 3z - 17 = 0$

40. $\bar{r}.(2\hat{\imath} + \hat{\jmath} + 2\hat{k}) + 3 = 0$ or

$\bar{r}.(-\hat{\imath} + 2\hat{\jmath} - 2\hat{k}) + 3 = 0$

41. (0, -1, -3)

42. $\left(-\frac{51}{7}, -\frac{18}{7}, \frac{43}{7}\right)$

43. $29x$ –27y–22z=85

44. 13

45. 1 unit

46. $x - 20y + 27z = 14$

47. $x - 2y + 2z = 0$ *and* $x - 2y + 2z = 6$

48. $\left(\frac{1}{2}, -\frac{1}{2}, -\frac{3}{2}\right)$

49. $\frac{1}{\sqrt{6}}$

50. $\frac{17}{2}$ units

51. $\bar{r}.(9\hat{\imath} + 17\hat{\jmath} + 23\hat{k}) = 20$

52. $2x - 7y + 4z + 15 = 0$

54. $x - 2y + z = 0$

55. $2\sqrt{29}$ units

56. $\frac{8}{\sqrt{29}}$

58. $\bar{r}.(\sqrt{2}\hat{\imath} + \hat{\jmath} + \hat{k}) = 2$

59. $(1, 2, 3), \sqrt{14}$

61. $7x + 13y + 4z = 9, \cos^{-1}\left(\frac{4}{\sqrt{234}}\right)$

62. $SD = 14\, units,$

$$\frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}$$

CHAPTER 13

PROBABILITY

POINTS TO REMEMBER

- **Conditional Probability:** If A and B are two events associated with any random experiment, then P(A/B) represents the probability of occurrence of event A knowing that event B has already occurred.

$$P\ (A/B) = \frac{P(A \cap B)}{P(B)}, P(B) \neq 0$$

P(B) ≠ 0, means that the event should not be impossible.

$$P(A \cap B) = P(A \text{ and } B) = P(B) \times P(A/B)$$

- Similarly P(A ∩ B ∩ C) = = P(A) × P(B/A) × P(C/AB)

P (A/S) = P(A), P(A/A)= 1, P(S/A)= 1, P(A^1/B)= 1-P (A/B)

- **Multiplication Theorem on Probability:** If the event A and B are associated with any random experiment and the occurrence of one depends on the other, then

$$P(A \cap B) = P(A) \times P(B/A) \text{ where } P(A) \neq 0$$

- When the occurrence of one does not depend on the other then these event are said to be independent events.

Here P(A/B) = P(A) and P(B/A) = P(B)

$$P(A \cap B) = P(A) \times P(B)$$

- Theorem on total probability: If $E_1, E_2, E_3 \ldots, E_n$ be a partition of sample space and $E_1, E_2 \ldots, E_n$ all have non-zero probability. A be any event associated with sample space S, which occurs with $E_1, or\ E_2, \ldots, or\ E_n$, then

$$P(A) = P(E_1) . P(A/E_1) + P(E_2) . P(A/E_2) + \ldots\ldots + P(E_n) . P(A/E_n)$$

If A & B are independent then (i) A ∩ B^c, (ii) A^c ∩ B& (iii) A^c ∩ b^c are also independent.

- **Bayes' theorem :** Let S be the sample space and $E_1, E_2 \ldots., E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A is any event which occurs with $E_1, or\ E_{2\ or} \ldots., E_n$, then

$$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{\sum_{i=0}^{n} P(E_i)P(A/E_i)}$$

- **Random variable :** It is real valued function whose domain is the sample space of random experiment.

- **Probability distribution :**It is a system of number of random variable (X), such that

X:	X_1	X_2	$X_{3\ldots.}$	X_n
P(X):	$P(X_{1)}$	$P(X_2)$	$P(X_{3)\ldots.}$	$P(X_{n)}$

Where P(x_i)> 0 and$\sum_{i=0}^{n} P(x_i) = 1$

- Mean or expectation of a random variables (X) is donated by E(X)

$$E(X) = \mu = \sum_{i=0}^{n} x_{i\ P(x_i)}$$

- Variance of X denoted by var(X) or σ_{x^2} and

$$Var(X) = \sigma_{x^2} = \sum_{i=0}^{n} (x_i - \mu)^2 P(X_i) = \sum_{i=0}^{n} x_i^2 P(x_i) - \mu^2$$

- The non-negative number $\sigma_x = \sqrt{var(X)}$ is called standard deviation of random variable X.

- **Bernoulli Trials:** Trials of random experiment are called Bernoulli trails if:

(xi) Number of trials is finite.

(xii) Trials are independent.

(xiii) Each trial has exactly two outcomes-either success or failure.

(xiv) Probability of success remain same in each trail.

- **Binomial distribution:**

  P(X = r) = n_{c_r} , $q^{n-r}p^r$, where r= 0, 1, 2, …n

  P = Probability of Success

  q = Probability of Failure

  n = total number of trails

  r = value of random variables.

Very Short Answer Type Question (1 Mark)

1. Find P (A/B) if P(A)= 0.4, P(B)= 0.8 and P (B/A)= 0.6

2. Find P (A ∩ B) if A and B are two events such that P(A) = 0.5, P(B) = 0.6 and P (A ∪ B) = 0.8

3. A soldier fires three bullets on enemy: The probability that the enemy will be killed by one bullet is 0.7. What is the probability that the enemy is still alive?

4. If P(A)= $\frac{1}{2}$, P(B)= $\frac{7}{12}$ and P (not A or not B)=$\frac{1}{4}$. State whether A and B are independent.

5. Three coins are tossed once. Find the probability of getting at least one head.

6. The probability that a student is not a swimmer is $\frac{1}{5}$. Find the probability that out of 5 students, 4 are swimmers.

7. Find P(A/B), if P(B) = 0.5 and P(A ∩ B) = 0.32

Short Answer Type Questions (2 Marks)

8. If A and B are two events such that $P(A) \neq 0$, then find P (B/A) if (i) A is a subset of B (ii) $A \cap B = \phi$

9. A random variable X has the following probability distribution find K.

X	0	1	2	3	4	5
P(X)	$\frac{1}{15}$	K	$\frac{15K-2}{15}$	K	$\frac{15K-1}{15}$	$\frac{1}{15}$

10. If $P(A) = \frac{1}{2}$, $P(A \cup B) = \frac{3}{5}$, and P(B) =q find the value of q if A and B are (i) Mutually exclusive (ii) independent events.

11. If $P(A) = \frac{3}{10}$, $P(B) = \frac{2}{5}$ and $P(A \cup B) = \frac{3}{5}$, then find P(B/A) + P(A/B)

12. A die is rolled if the out come is an even number. What is the probability that it is a prime?

13. If A and B are two-events such that $P(A) = \frac{1}{4}$, $P(B) = \frac{1}{2}$ and $P(A \cap B) = \frac{1}{8}$ Find P (not A and not B).

14. A pair of die is rolled six times. Find the probability that a third sum of 7 is observed in sixth throw.

15. Ten eggs are drawn successively with replacement from a lot containing 10% defective eggs. Find the probability that there is at least one defective egg.

SHORT ANSWER TYPE QUESTIONS (4 MARKS)

16. A problem in mathematics is given to three students whose chances of solving it are $\frac{1}{2}, \frac{1}{3}$and$\frac{1}{4}$. What is the probability that the problem is solved?

17. Two aeroplanes X and Y bomb a target in succession. There probabilities to hit correctly are 0.3 and 0.2 respectively. The second plane will bomb only if first miss the target. Find the probability that target is hit by Y plane.

18. A can hit a target 4 times in 5 shots B three times in 4 shots and C twice in 3 shots. They fire a volley. What is the probability that at least two shots hit?

19. Two dice are thrown once. Find the probability of getting an even number on the first die or a total of 8.

20. A and B throw a die alternatively till one of them throws a '6' and wins the game. Find their respective probabilities of winning, if A starts the game.

21. A man takes a step forward with probability 0.4 and backward with probability 0.6. Find the probability that at the end of eleven steps he is one step away from the starting point.

22. Two cards are drawn from a pack of well shuffled 52 cards one by one with replacement. Getting an ace or a spade is considered a success. Find the probability distribution for the number of successes.

23. In a game, a man wins a rupee for a six and looses a rupee for any other number when a fair die is thrown. The man decided to throw a die thrice but to quit as and when he gets a six. Find the expected value of the amount he wins/looses.

24. Suppose that 10% of men and 5% of women have grey hair. A grey haired person is selected at random. What is the probability that the selected person is male assuming that there are 60% males and 40% females?

25. A card from a pack of 52 cards is lost. From the remaining cards of the pack, two cards are drawn. What is the probability that they both are diamonds?

26. If A and B are two independent events such that $P(\overline{A} \cap B) = \frac{2}{15}$ and $P(A \cap \overline{B}) = \frac{1}{6}$ then find P(A) and P(B).

Long Answer Type Questions (6 Marks)

27. In a hurdle race, a player has to cross 8hurdles. The probability that he will clear a hurdle is $\frac{4}{5}$, what is the probability that he will knock down in fewer than 2 hurdles?

28. Bag A contains 4 red, 3 white and 2 black balls. Bag B contains 3 res, 2 white and 3 black balls. One ball is transferred from bag A to bag B and then a ball is drawn from bag B. The ball so drawn is found to be red. Find the probability that the transferred ball is black.

29. If a fair coin is tossed 10 times, find the probability of getting.

(i) Exactly six heads,

(ii) at least six heads,

(iii) at most six heads.

30. A doctor is to visit a patient. From the past experience, it is known that the probabilities that he will come by train, bus, scooter by other means of transport are respectively $\frac{3}{10}, \frac{1}{5}, \frac{1}{10}$ *and* $\frac{2}{5}$ .The probabilities that he will be late are $\frac{1}{4}, \frac{1}{3}$ *and* $\frac{1}{12}$ if he comes by train, bus and scooter respectively but if comes by other means of transport, then he will not be late. When he arrives, he is late. What is the probability that he comes by train?

31. A man is known to speak truth 3 out of 4 times. He throws a die and reports that it is six. Find the probability that it is actually a six.

32. An insurance company insured 2000 scoter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident is 0.01, 0.03 and 0.15 respectively. One of the insured persons meets with an accident. What is the probability that he is a scooter driver?

33. Three cards from a pack of 52 cards are lost. One card is drawn from the remaining cards. If drawn card is heart, find the probability that the lost cards were all hearts.

34. A box X contains 2 white and 3 red balls and a bag Y contains 4 white and 5 red balls. One ball is drawn at random from one of the bags and is found to be red. Find the probability that it was drawn from bag Y.

35. In answering a question on a multiple choice, a student either knows the answer or guesses. Let $\frac{3}{4}$ be the probability that he knows the answer and $\frac{1}{4}$ be the probability that he guesses. Assuming that a student who guesses at the answer will be incorrect with probability $\frac{1}{4}$. What is the probability that the student knows the answer, given that he answered correctly?

36. Suppose a girl throws a die. If she gets 5 or 6, she tosses a coin three times and notes the number of heads. If she gets 1, 2, 3 or 4 she tosses a coin once and notes whether a head or tail is obtained. If she obtained exactly one head. What is the probability that she throws 1, 2, 3 or 4 with the die?

37. In a bolt factory machines, A, B and C manufacture bolts in the ratio 6:3:1. 2%, 5% and 10% of the bolts produced by them respectively are defective. A bolt is picked up at random from the product and is found to be defective. What is the probability that it has been manufactured by machine A?

38. Two urns A and B contain 6 black and 4 white, 4 black and 6 white balls respectively. Two balls are drawn from one of the urns. If both the balls drawn are white, find the probability that the balls are drawn from urn B.

39. Two cards are drawn from a well shuffled pack of 52 cards. Find the mean and variance for the number of face cards obtained.

40. A letter is known to have come from TATA NAGAR or from CALCUTTA on the envelope first two consecutive letters 'TA' are visible. What is the probability that the letter come from TATA NAGAR?

41. Two groups are competing for the position on the Board of Directors of a corporation. The probabilities that first and the second group will win are 0.6 and 0.4 respectively. Further if the first group wins, the probability of introducing a new product is 0.7 and the corresponding probability is 0.3 if the second group wins. Find the probability that the new product introduced was by the second group.

42. Two numbers are selected at random (without replacement) from positive integers 2, 3, 4, 5, 6, 7. Let X denotes the larger of the two numbers obtained. Find the mean and variance of the probability distribution of X.

43. An urn contains five balls. Two balls are drawn and are found to be white. What is the probability that all the balls are white?

44. Find the probability distribution of the number of doublets in four throws of a pair of dice. Also find the mean and S.D. of his distribution.

45. Three critics review a book. Odds in favour of the book are 5:2, 4:3 and 3:4 respectively for the three critics. Find the probability that the majority are in favour of the book.

46. A box contains 2 Black, 4 White and 3 Red balls. One by one all balls are drawn without replacement and arranged in sequence of drawing. Find the probability that the drawn balls are in sequence of BBWWWRRR.

47. A bag contains 3 White, 3Black and 2 Red balls. 3 balls are successively drawn without replacement. Find the probability that third ball is red.

Answers

1. 0.3
2. $\frac{3}{10}$
3. $(0.3)^3$
4. No
5. $\frac{7}{8}$
6. $\left(\frac{4}{5}\right)^4$
7. $\frac{16}{25}$
8. (i) 1 (ii) 0
9. K = $\frac{4}{15}$
10. (i) $\frac{1}{10}$ (ii) $\frac{1}{5}$
11. $7/12$
12. $1/3$
13. $3/8$
14. $1250 \times \left(\frac{1}{6}\right)^6$

15. $1-\left(\frac{9}{10}\right)^{10}$

16. $\frac{3}{4}$

17. $\frac{7}{22}$

18. $\frac{5}{6}$

19. $\frac{5}{9}$

20. $\frac{6}{11}, \frac{5}{11}$

21. 0.3678 or $11_{C_5}(0.4)^5(0.6)^5$

22.

X	0	1	2
P(X)	81/169	72/169	16/169

23. $-\frac{91}{54}$

24. $\frac{3}{4}$

25. $\frac{11}{50}$

26. $P(A)=\frac{1}{5}$, $P(B)=\frac{1}{6}$ or $P(A)=5/6$, P(B)= 4/5

27. $\frac{12}{5}\left(\frac{4}{5}\right)^7$

28. $\frac{6}{31}$

29. $(i)\ \frac{105}{512}$ (ii)$\frac{193}{512}$ (iii)$\frac{53}{64}$

30. $\frac{1}{2}$

31. $\frac{3}{8}$

33. $\frac{10}{49}$

34. $\frac{25}{52}$

35. $\frac{12}{13}$

36. $\frac{8}{11}$

37. $\frac{12}{37}$

38. $\frac{5}{7}$

39. Mean =$\frac{6}{13}$ Variance = $\frac{60}{169}$

40. $\frac{7}{11}$

41. $\frac{2}{9}$

42. Mean =$\frac{17}{3}$ Variance= $\frac{14}{9}$

43. $\frac{1}{2}$

44. Mean = 2/3 S.D.= $\frac{\sqrt{5}}{3}$

45. $\frac{209}{343}$

46. $\frac{1}{1260}$

47. $\frac{1}{4}$